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TRIANGULAR – PENTAGONAL NUMBERS AND INTERESTING PUZZLE

Dr. R. Sivaraman
Associate Professor, Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai, Tamilnadu

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Abstract:

Among several interesting types of numbers that exist in mathematics, polygonal numbers are so special because they provide exotic connections between number theory and geometry. Triangular and Pentagonal numbers are special types of more general Polygonal numbers. In this paper, I will introduce these numbers and determine numbers which are both triangular and pentagonal. I will also solve a related interesting puzzle.

Key Words: Triangular Numbers, Pentagonal Numbers, Continued Fraction, Convergent, Recurrence Relations

1. Introduction:

The study of polygonal numbers has been done for more than two millennia. The properties exhibited by polygonal numbers continue to fascinate mathematicians of all generations. Triangular and Pentagonal numbers are special cases of polygonal numbers. In this paper, I will formally introduce triangular and pentagonal numbers and determine set of numbers which are simultaneously triangular and pentagonal using a novel method involving continued fraction. The same continued fraction is used to solve an interesting puzzle regarding square numbers.

2. Definition:

The *n*th polygonal number of order *k* where $k \geq 3$ is defined by

$$P_k(n) = \frac{n}{2}\left[(k-2)n-(k-4)\right] \quad (1)$$

In particular, if $k = 3$, then the polygonal numbers become triangular numbers. In view of (1), the *n*th triangular number is given by $P_3(n) = \frac{n(n+1)}{2}$ (2)

Similarly, if $k = 5$, then the polygonal numbers become pentagonal numbers. In view of (1), the *n*th pentagonal number is given by $P_5(n) = \frac{n(3n-1)}{2}$ (3)

In the following section, I will discuss the computation of determining the equality of numbers of the form given by expressions (2) and (3).

3. Triangular – Pentagonal Numbers:

To determine numbers which are simultaneously triangular and pentagonal, we wish to find positive integers *n*, *m* such that $P_5(n) = P_3(m)$.

That is, we need to find *n*, *m* such that $\frac{n(3n-1)}{2} = \frac{m(m+1)}{2}$ (4)

Multiplying both sides of (4) by 24 and adding 1 on both sides, we get

$$36n^2 - 12n + 1 = 12m^2 + 12m + 1 \Rightarrow (6n-1)^2 = 3(2m+1)^2 - 2$$

We thus obtain $(6n-1)^2 - 3(2m+1)^2 = -2$ (5)

Now if $x = 6n-1, y = 2m+1$ then equation (5) takes the form $x^2 - 3y^2 = -2$ (6)

To solve (6), we first notice that $(1-\sqrt{3}) \times (1+\sqrt{3}) = -2$

From this equation, we observe the following computations

$$1-\sqrt{3} = \frac{-2}{1+\sqrt{3}} = \frac{-2}{2-(1-\sqrt{3})} = \frac{-2}{2+\frac{2}{2-(1-\sqrt{3})}} = \frac{-2}{2+\frac{2}{2+\frac{2}{2-(1-\sqrt{3})}}} = \frac{-2}{2+\frac{2}{2+\frac{2}{2+\frac{2}{2+\cdots}}}}$$

$$\sqrt{3}=1+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cdots}}}} \quad (7)$$

We can now use (7) to determine all positive integer solutions of (6). For this, we compute the successive convergents of the continued fraction expression obtained in (7). The sequence of successive convergents are given by

$$\frac{1}{1},1+\frac{2}{2}=\frac{2}{1},1+\cfrac{2}{2+\cfrac{2}{2}}=1+\frac{2}{3}=\frac{5}{3},1+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2}}}=1+\cfrac{2}{2+\cfrac{2}{3}}=1+\frac{3}{4}=\frac{7}{4},1+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2}}}}=1+\cfrac{2}{2+\cfrac{3}{4}}=1+\frac{8}{11}=\frac{19}{11},$$

$$1+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2}}}}}=1+\cfrac{2}{2+\cfrac{8}{11}}=1+\frac{11}{15}=\frac{26}{15},1+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2}}}}}}=1+\cfrac{2}{2+\cfrac{11}{15}}=1+\frac{30}{41}=\frac{71}{41},$$

$$1+\cfrac{2}{2+\cfrac{30}{41}}=1+\frac{41}{56}=\frac{97}{56},1+\cfrac{2}{2+\cfrac{41}{56}}=1+\frac{112}{153}=\frac{265}{153},1+\cfrac{2}{2+\cfrac{112}{153}}=1+\frac{153}{209}=\frac{362}{209},1+\cfrac{2}{2+\cfrac{153}{209}}=1+\frac{418}{571}=\frac{989}{571},$$

If we now list the first, third, fifth, seventh, ninth, eleventh convergents obtained above then we get $\frac{1}{1},\frac{5}{3},\frac{19}{11},\frac{71}{41},\frac{265}{153},\frac{989}{571},\ldots$. Among these fractions if we consider the numerators as *x* and denominators as *y* then we observe that $(x,y)=(1,1);(5,3);(19,11);(71,41);(265,153);(989,571);$ are solutions of (6) namely $x^2-3y^2=-2$.

Similarly if we consider the numerators as *x* and denominators as *y* in the second, fourth, sixth, eighth, tenth convergents then we observe that $(x,y)=(2,1);(7,4);(26,15);(97,56);(362,209);$ are solutions to the Pell's equation $x^2-3y^2=1$. If we consider the ratios $\frac{x}{y}$ from the above pairs namely $\frac{2}{1},\frac{7}{4},\frac{26}{15},\frac{97}{56},\frac{362}{209},\ldots$ then they provide best possible approximates to $\sqrt{3}$.

Moreover, for both cases the subsequent values of *x* and *y* can be obtained using the recurrence relations $x_{n+2}=4x_{n+1}-x_n, y_{n+2}=4y_{n+1}-y_n$ (8)

Now in our conversion of equation (5) to (6), since $n=\frac{x+1}{6}, m=\frac{y-1}{2}$ and *n*, *m* are positive integers, the pairs $(x,y)=(5,3);(71,41);(989,571);(13775,7953);(191861,110771);\ldots$ (9) provide solutions to (4). We notice that except for the first two pairs the subsequent pairs in (9) can be obtained recursively through $x_{n+2}=14x_{n+1}-x_n, y_{n+2}=14y_{n+1}-y_n$ (10).

Thus for the respective pairs listed in (9), the corresponding values of (n,m) are given by

$$(n,m)=(1,1);(12,20);(165,285);(2296,3976);(31977,55385);\ldots \quad (11)$$

Thus using (4) and (11), the Triangular – Pentagonal numbers are given by

1, 210, 40755, 7906276, 1533776805, … (12)

Hence (12) provides the first five triangular – pentagonal numbers.

4. Puzzle:

In this section, I will pose a puzzle and provide its solutions using the results obtained in previous section in determining triangular – pentagonal numbers.

The puzzle is the following: Determine all positive integers *k* such that both *k* + 1 and 3*k* + 1 are perfect squares (13)

Solution:

Let $k+1=y^2, 3k+1=x^2$. Then we get $x^2-3y^2=-2$. But this is exactly equation (6). Thus $(x,y)=(1,1);(5,3);(19,11);(71,41);(265,153);(989,571);$ are solutions of $x^2-3y^2=-2$.

Therefore the possible values of *k* are given by $k=y^2-1=\frac{x^2-1}{3}$. Hence the solutions to (13) are given by the numbers 8, 120, 1680, 23408, 326040, . . . (14)

5. Conclusion:

In this paper, I had defined triangular and pentagonal numbers which were special cases of more general class of numbers called polygonal numbers. The main objective of the paper is to describe positive integers which are simultaneously triangular and pentagonal numbers. In this attempt, I ended with solving equation (6) of the form $x^2-3y^2=-2$.

Using a novel idea of computing continued fraction for $\sqrt{3}$ derived in (7) and considering its alternate convergents, I had obtained solutions not only to the equation $x^2-3y^2=-2$ but also to the Pell's equation $x^2-3y^2=1$. We notice that the successive solutions to both the equations satisfy nice reccurence relations given by (8). Using the ordered pairs given in (9), we can obtain the list of numbers which are triangular as well as pentagonal numbers. Interestingly, the pairs described in (9) helping us to yield the required numbers also satisfy nice recurrence relations given by (10). The pairs in (11) provides the possible values of *n*, *m* which I have substituted in (4) to obtain the first five triangular – pentagonal numbers as described in (12). One can extend this procedure and generate as many as possible numbers which are both triangular and pentagonal. Though the idea of generating triangular – pentagonal numbers was well known, in this paper, I had used a novel method of computing the same using nice continued fraction expansion derived in (7).

Finally, an interesting puzzle given by (13) was discussed in section 4. While solving the puzzle, we saw that it reduces to the equation $x^2-3y^2=-2$ which is exactly the same to obtain triangular – pentagonal numbers. Hence, using those solutions, I had solved the puzzle and obtained its solutions as given in (14). Thus solving a simple equation through a novel continued fraction expansion has helped us to generate triangular – pentagonal numbers as well as solving a related interesting puzzle. Similar ideas can be employed to solve other types of interesting problems.

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